The Modeling of Small Scales in Two-Dimensional Turbulent Flows: A Statistical Mechanics Approach

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In previous work we have defined statistical equilibrium states for 2D incompressible Euler equations. We study here the relaxation process toward equilibrium. This leads to a natural modeling of the small scales in turbulent flows, which might be relevant for meteorological and oceanographic applications. Numerical simulations illustrating the performance of these new models are presented.

1. INTRODUCTION

It is well known that direct numerical simulations of 2D turbulent flows need a mesh size of the order of the viscous dissipation scale. This causes a drastic limitation in the Reynolds numbers that can be reached. In practice, this difficulty is overcomed by introducing an artificial turbulent diffusion term in the equations in order to limit at a reasonable level the number of degrees of freedom of the system. This empirical recipe rests on the belief that we can parametrize in a simple way the statistical effect of the small scales on the large-scale motion (in which we are actually interested). This assumption is a general prerequisite for the feasibility of large-eddy simulation (see, for example, Sadourny[28] and Basdevant and Sadourny[2]).

The expression "turbulent diffusion" suggests that turbulence is producing, by an irreversible process, some kind of disorder or entropy.

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The recent development of a statistical equilibrium theory for the perfect fluid[16-19, 25, 26] and the related numerical and experimental studies[7, 8, 29-31] support this view. Indeed we explicitly know the entropy functional that the turbulent motion increases.

The purpose of this paper is to show how the statistical equilibrium theory can be used to parametrize the effect of the small scales. This question was first investigated by Robert and Sommeria,[27] who proposed a set of relaxation equations leading the system toward equilibrium. We give here a further analysis of the relaxation mechanism which helps us to understand the limitations of the first approach and yields a more involved expression for the turbulent diffusion terms. Our relaxation equations are of a diffusion-convection type. The main difference from Navier–Stokes equations is that they conserve the energy and all the constants of the motion of Euler equations.

In Section 2 we review the main features of the theory of equilibrium states for 2D Euler equations. In Section 3 we discuss numerical and experimental tests of the equilibrium theory, pointing out the main features of the relaxation process. We establish our relaxation equations and give an insight into the relation between the asymptotic behavior of the relaxation equations and Gibbs states. In Section 4, to show the relevance of this model, we compare the results of numerical simulations using our relaxation equations with direct numerical simulations using Navier–Stokes equations at large Reynolds number.

2. STATISTICAL EQUILIBRIUM THEORY FOR 2D TURBULENT FLOWS

This section is devoted to a brief summary of the equilibrium theory. We refer to refs. 16, 25, and 26 for more detailed presentations.

2.1. Perfect Fluid

2.1.1. Euler Equations. We start here with Euler equations in an open, bounded, simply connected, and regular domain Ω of the plane. Let $\mathbf{u}(t, \mathbf{x})$ be the velocity field; the incompressibility condition is solved by introducing the stream function $\psi(t, \mathbf{x})$. We consider the scalar vorticity $\omega(t, \mathbf{x}) = (\nabla \times \mathbf{u}) \cdot \mathbf{e}_3$, with $\mathbf{e}_3$ the unit vector normal to the plane, and we write the Euler system

$$\omega_t + \nabla \cdot (\omega \mathbf{u}) = 0, \qquad \omega(0, \mathbf{x}) = \omega_0(\mathbf{x}) \tag{1}$$

$$\mathbf{u} = \nabla \times (\psi \mathbf{e}_3), \qquad \omega = -\Delta\psi, \qquad \psi = 0 \quad \text{on} \quad \partial\Omega \tag{2}$$

For any bounded initial vorticity $\omega_0(\mathbf{x})$ the system (1), (2) has a unique bounded solution $\omega(t, \mathbf{x})$.[33] We assume first that the initial condition is made of patches with n uniform vorticity levels a_i. Then, all the known constants of the motion in our domain are the following functionals:

The energy

$$E(\omega) = \tfrac{1}{2} \int_{\Omega} \mathbf{u}^2 \, d\mathbf{x} = \tfrac{1}{2} \int_{\Omega} \psi\omega \, d\mathbf{x}$$

The area $|\Omega^i|$ of each vorticity patch Ω^i with uniform value a_i.

And if Ω is a disk centered at 0, $B(0, R)$, the angular momentum with respect to 0:

$$M(\omega) = \int_{\Omega} \mathbf{x} \times \mathbf{u}(\mathbf{x}) \, d\mathbf{x} = \left[\tfrac{1}{2} \int_{\Omega} (R^2 - \mathbf{x}^2) \, \omega(\mathbf{x}) \, d\mathbf{x} \right] \mathbf{e}_3$$

2.1.2. Macroscopic Equilibrium States.

After some evolution, the solution of the Euler equations becomes in general extremely complicated. Instead of a detailed description of the vorticity field, we introduce the macroscopic variables $p_i(\mathbf{x})$, $i = 1,..., n$, which give at each point $\mathbf{x}$ the probability of finding the value a_i. Taking the thermodynamic limit of a family of approximate Liouville measures for (1), (2), it has been proved[16, 25] that "an overwhelming majority" of all the vorticity fields with given constants of the motion are close to a macroscopic state (the equilibrium state) or to a set of such states (the equilibrium set). These states are obtained by maximizing the mixing entropy:

$$S(\mathbf{p}) = \int_{\Omega} s(\mathbf{p}(\mathbf{x})) \, d\mathbf{x}$$

$$\mathbf{p}(\mathbf{x}) = (p_1(\mathbf{x}),..., p_n(\mathbf{x})), \qquad s(\mathbf{p}) = -\sum_i p_i \operatorname{Ln} p_i$$

under the following constraints:

(i) $\sum_i p_i(\mathbf{x}) = 1$ for all $\mathbf{x}$.

(ii) $F_i(\mathbf{p}) = \int_{\Omega} p_i(\mathbf{x}) \, d\mathbf{x} = |\Omega^i|$, $i = 1,..., n$.

(iii) $E(\sum a_i p_i) = E(\omega_0)$.

It was shown that this variational problem, to which we will refer as (VP) in the following, always has a solution (possibly not unique).[25, 26, 29] If $\mathbf{p}^* = (p_1^*,..., p_n^*)$ is a solution of (VP) such that each function $p_i^*(\mathbf{x})$ is

continuous and strictly positive on Ω, we can show that there exist
Lagrange multipliers $\alpha_1,..., \alpha_n, \beta$ such that

$$p_i^*(\mathbf{x}) = \frac{\exp(-\alpha_i - \beta a_i \psi^*(\mathbf{x}))}{Z(\psi^*(\mathbf{x}))}, \qquad i = 1,..., n \qquad (3)$$

where $Z(\psi) = \sum_i \exp(-\alpha_i - \beta a_i \psi)$, and ψ^* is the stream function
associated with the locally averaged vorticity $\bar{\omega}_* = \sum_i a_i p_i^*$. As a result of
the relationship $\sum p_i(\mathbf{x}) = 1$, the functionals $F_1,..., F_n$ give only $n - 1$ inde-
pendent constraints, and we can always take $\alpha_n = 0$. Thus to find the equi-
librium states, we must solve the nonlinear elliptic equation (equation of
Gibbs states)[25, 26, 29]

$$-\varDelta\psi = -\frac{1}{\beta}\frac{d}{d\psi}\operatorname{Ln} Z, \qquad \psi = 0 \quad \text{on} \quad \partial\Omega \qquad (4)$$

It always has a unique solution when β is greater than some negative value
β_c, but when $-\beta$ is sufficiently large, bifurcations to multiple solutions
generally occur.[29]

2.1.3. General Case. The extension to the general case of any
bounded vorticity function $\omega_0(\mathbf{x})$ is straightforward. We define:

- The probability distribution π_0 on $\mathbb{R}$ given by

$$\int f(z) \, d\pi_0(z) = \frac{1}{|\Omega|} \int_\Omega f(\omega_0(\mathbf{x})) \, d\mathbf{x} \qquad \text{for all} \quad f$$

- The macroscopic description of the flow, which is given at $\mathbf{x}$ by the
probability distribution $\rho(\mathbf{x}, z) \, \pi_0$.

- The entropy functional:

$$S(\rho) = -\int_\Omega d\mathbf{x} \int \rho \operatorname{Ln} \rho \, d\pi_0(z)$$

which we have to maximize under the following constraints:

(iv) $\int \rho(\mathbf{x}, z) \, d\pi_0(z) = 1, \forall \mathbf{x}$.
(v) $\int_\Omega \rho(\mathbf{x}, z) \, \pi_0 \, d\mathbf{x} = |\Omega| \, \pi_0$.
(vi) $E(\bar{\omega}) = E(\omega_0)$, where $\bar{\omega}(\mathbf{x}) = \int z\rho(\mathbf{x}, z) \, d\pi_0(z)$.

This yields a similar expression for the Gibbs states:

$$\rho^*(\mathbf{x}, z) = \frac{\exp(-\alpha(z) - \beta z\psi^*(\mathbf{x}))}{Z(\psi^*(\mathbf{x}))} \qquad (3')$$

with $Z(\psi) = \int \exp(-\alpha(z) - \beta z \psi) \, d\pi_0(z)$, β is the Lagrange multiplier of the energy constraint, the continuous function $\alpha(z)$ is now associated with the (infinite) set of constraints (v), and ψ^* is the stream function of $\bar{\omega}_*$. The associated Gibbs state equation takes the same form (4).

2.2. Slightly Viscous Fluid

Let us suppose that we observe a real (viscous) flow on some time interval during which it gets close to some inertial equilibrium. We first notice that the distribution of vorticity is immediately altered: for example, if ω_0 is a vortex patch, it will be at once smoothed into a continuous distribution. Thus we shall suppose that π_0 is diffuse. Also, it is well known that, for n large, the functionals $\Gamma_n = \int_\Omega \omega(t, \mathbf{x})^n \, d\mathbf{x}$ are quickly dissipated, while a finite number of them may be approximately conserved on the time interval of interest.

By the slightly viscous case we mean the situation where a turbulent flow gets close to an equilibrium in a time interval during which the energy and a finite number $\Gamma_1, ..., \Gamma_N$ of these functionals are approximately conserved.

Then it is natural to associate with such a flow the equilibrium state given by the maximization of the entropy $S(\rho)$ under the constraints (iv), (vi), and

(v') $\quad \int_\Omega d\mathbf{x} \int z^n \rho(\mathbf{x}, z) \, d\pi_0(z) = \Gamma_n, \, n = 1, ..., N.$

This yields a mean-field (Gibbs state) equation of the form (4), with

$$Z(\psi) = \int \exp\left(-\sum \alpha_n z^n - \beta z \psi\right) d\pi_0(z)$$

where $\alpha_1, ..., \alpha_N$ are the Lagrange multipliers associated with the constraints (v').

3. RELAXATION TOWARD THE EQUILIBRIUM

3.1. Tests of the Equilibrium Theory

The relevance of this equilibrium theory was tested by experiments[7, 8] and numerical simulations using Navier–Stokes equations at large Reynolds number.[29, 31] The case of an initial vortex patch is studied in refs. 7, 8, and 29, while a vorticity function with three levels is considered in ref. 31. In all these tests we consider situations where a turbulent flow converges toward some equilibrium state, and we compare the relationship

$\bar{\omega} = f(\psi)$ that we get (at equilibrium) in experiments or numerical simulations with the predictions of the theory. An analysis of these tests leads us to the following important remarks. The first is that, in order to provide a relevant comparison, the experimental or simulated flow must go toward an equilibrium quickly enough, otherwise the viscosity can dramatically alter the constants of the motion and cause great change in the final state. In practice it is not very hard to find situations reasonably satisfying this requirement. The second observation is that the theoretical predictions are fairly well confirmed in the regions where a turbulent mixing occurred. If there are also quiet regions, the theoretical relationship is not satisfied on the whole domain. We can observe in many cases that the flow converges toward some stationary state which locally satisfies the Gibbs state relationship given by the theory, but is not a maximum entropy state on the whole domain. We can give an intuitive view of this phenomenon. It is a well-known general fact in thermodynamics that the presence of fluctuations is necessary to drive a system toward its statistical equilibrium. Here the fluctuations of the system are given by the small-scale oscillations of the vorticity field. But it may occur that the fluctuations vanish before the system is close to its global equilibrium, so that the system appear to be frozen in a state which is not the statistical equilibrium state on the whole domain. The main purpose of this section is to give a more precise content to this heuristic picture.

3.2. Relaxation Process for a Perfect Fluid

3.2.1. The *n*-Level Case, a Maximum Entropy Production Principle.

Let us first consider the case where $\omega_0(\mathbf{x})$ takes only n distinct values $a_1,...,a_n$. We shall assume that during its evolution toward a final equilibrium state, the system can already be described macroscopically in terms of a set of local probabilities $p_1(t, \mathbf{x}),..., p_n(t, \mathbf{x})$. In other words, the system has already undergone fine-scale vorticity oscillations. The locally averaged vorticity is $\bar{\omega}(t, \mathbf{x}) = \sum_i a_i p_i(t, \mathbf{x})$ and $\bar{\mathbf{u}}$ is the corresponding velocity field obtained by the integration of (2) where ω is replaced by $\bar{\omega}$. The vorticity patches are transported by $\bar{\mathbf{u}}$, and we suppose that in addition they undergo a diffusion process, so that the conservation equation for each vorticity probability can be written

$$(p_i)_t + \nabla \cdot (p_i \bar{\mathbf{u}} + \mathbf{J}_i) = 0, \qquad i = 1,..., n \qquad (5)$$

$\mathbf{J}_i$ is the diffusion current of the patch i. We impose the boundary condition $\mathbf{J}_i \cdot \mathbf{n} = 0$, so that the total area occupied by each patch is conserved. We can assume (without loss of generality) $\sum \mathbf{J}_i = \mathbf{0}$, i.e., the locally averaged velocity of the fluid is $\bar{\mathbf{u}}$. We denote $\mathbf{J}_\omega = \sum_i a_i \mathbf{J}_i$.

We shall assume that the kinetic energy associated with the diffusion currents $\mathbf{J}_i$:

$$E_d(\mathbf{J}_1,...,\mathbf{J}_n) = \frac{1}{2}\int_\Omega \sum \frac{\mathbf{J}_i^2}{p_i}\,d\mathbf{x}$$

is small compared to the macroscopic kinetic energy:

$$E(\bar{\omega}) = \frac{1}{2}\int_\Omega \psi\bar{\omega}\,d\mathbf{x}$$

Let us now compute the rate of change of the energy E and the entropy S in the convection-diffusion process (5). Straightforward computations give

$$\dot{E} = \int_\Omega \nabla\psi \cdot \mathbf{J}_\omega\,d\mathbf{x}$$

$$\dot{S} = -\int_\Omega \sum_i \nabla \mathrm{Ln}\, p_i \cdot \mathbf{J}_i\,d\mathbf{x}$$

To get a closed set of equations, we need to relate the currents $\mathbf{J}_i$ to the probability field p_i. With this aim, we shall use two different arguments. First we shall exploit our knowledge of the entropy functional, which must increase. Second we shall also exploit an analysis of the dynamics: fine-scale oscillations of vorticity create local fluctuations in the velocity field which in turn induce a diffusion current for the mean vorticity (this will be detailed later). It appears that combining these two arguments fortunately yields a tractable expression for $\mathbf{J}_i$. To carry out the first part of our program, let us state a principle of general scope.

3.2.2. Maximum Entropy Production Principle (MEPP).
During the relaxation process toward the equilibrium, the system tends to maximize its rate of entropy production while it satisfies all the constraints imposed by the dynamics.

At first sight such a vague formulation seems trivial and useless, since there are generally untractable difficulties in expressing all the constraints imposed by the dynamics. In fact, as we shall see, the MEPP can be very efficient if we have a precise recipe to use it. The idea is to move forward by trial and error. We begin by guessing some crude set of constraints that the dynamics puts on the currents $\mathbf{J}_i$. The principle then yields a corresponding set of relaxation equations, which we can compare with the actual behavior of the system; so that we can learn something from the system, and deduce more realistic constraints, etc. In this approach the

MEPP is a very efficient tool for extracting some of the main features of the very intricate detailed dynamical behavior of the system.

Remark. Our MEPP must not be confused with the minimum entropy production principle of Prigogine. The latter, which is a reformulation of conservation laws, applies to a stationary state in the linear regime. We refer to ref. 12 for a detailed discussion on the subject. Although it was not explicitly formulated by this author, our principle is clearly in the spirit of Jaynes' ideas.[12]

Let us now apply our MEPP.

• The first evident dynamical constraint on the J_i is given by the conservation of the macroscopic energy: that is, $\dot{E}(J_1,..., J_n) = 0$.

• Also it is clear that at any point $\mathbf{x}$ the density of energy associated to the diffusion transport $\frac{1}{2} \sum (J_i^2/p_i)$ cannot be arbitrarily large. More precisely, we shall suppose that for any given state of the system there is a function $C(\mathbf{x})$ such that the following constraint holds:

$$\frac{1}{2} \sum \frac{J_i^2}{p_i}(\mathbf{x}) \leqslant C(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x}$$

Then the MEPP gives the following variational problem (VP.1):

$$\dot{S}(J_1,..., J_n) = \max \dot{S}(\mathscr{J}_1,..., \mathscr{J}_n)$$

under the following constraints:

(C1) $\sum_i \mathscr{J}_i(\mathbf{x}) = 0$ for all $\mathbf{x}$.

(C2) $\dot{E}(\mathscr{J}_1,..., \mathscr{J}_n) = 0$.

(C3) $\frac{1}{2} \sum (\mathscr{J}_i^2/p_i)(\mathbf{x}) \leqslant C(\mathbf{x})$ for all $\mathbf{x}$.

It can be shown that this problem always has a solution $J_1,..., J_n$ satisfying (C3) with equality (this mathematical issue is discussed in Appendix B); moreover, there exist a parameter β and a measurable function $A(\mathbf{x})$ such that

$$J_i = -A(\mathbf{x})[\nabla p_i - \beta(\bar{\omega} - a_i) p_i \nabla \psi] \tag{6}$$

This gives a general form for the currents $J_1,..., J_n$. The variable $(\geqslant 0)$ diffusion coefficient $A(\mathbf{x})$ is not known; but once it is given, the parameter β, which is the Lagrange multiplier of $\dot{E}$, is determined by the conservation of the energy:

$$\int_\Omega \nabla \psi \cdot J_\omega \, d\mathbf{x} = 0$$

which gives

$$\beta = -\int_{\Omega} \nabla\psi \cdot \nabla\tilde{\omega} A(\mathbf{x})\, d\mathbf{x} \Big/ \int_{\Omega} (\nabla\psi)^2\, (\overline{\omega^2} - \bar{\omega}^2)\, A(\mathbf{x})\, d\mathbf{x} \tag{7}$$

where $\overline{\omega^2} = \sum_i a_i^2\, p_i$.

Next we need to determine the diffusion coefficient $A(\mathbf{x})$. With this aim, let us consider the particular case $\beta = 0$. In this case, the energy constraint is not active and $\mathbf{J}_i$ is merely an ordinary diffusion current. Thus we can compute $A(\mathbf{x})$ by using the analogy with the convection-diffusion of a passive scalar. Let us introduce the notations

$$\omega = \bar{\omega} + \tilde{\omega}, \qquad \mathbf{u} = \bar{\mathbf{u}} + \tilde{\mathbf{u}}$$

$\tilde{\omega}$ is the fine-scale fluctuation of the vorticity and $\tilde{\mathbf{u}}$ the corresponding fluctuation of velocity. Let us suppose that some scalar density $\rho(t, \mathbf{x})$ is convected by the microscopic flow $\mathbf{u}$. Then the mean value $\bar{\rho}(t, \mathbf{x})$ of $\rho(t, \mathbf{y})$ on some ball $B(\mathbf{x}, r)$ is convected by the mean field $\bar{\mathbf{u}}$ and undergoes a diffusion process created by the fluctuation $\tilde{\mathbf{u}}$. Then $\bar{\rho}(t, \mathbf{x})$ will satify an equation of convection-diffusion type:

$$\bar{\rho}_t + \nabla \cdot (\bar{\rho}\bar{\mathbf{u}} + \mathbf{J}) = 0$$

where the diffusion current $\mathbf{J}$ can be calculated by classical methods (see Appendix A): $\mathbf{J} = -\mathbf{D}\,\nabla\bar{\rho}$, with the diffusion matrix

$$\mathbf{D}_{ij} = \text{const} \cdot \langle \tilde{u}_i, \tilde{u}_j \rangle = c\varepsilon^2\, \text{Ln}\left(\frac{r}{\varepsilon}\right) (\overline{\omega^2} - \bar{\omega}^2)\, \delta_{ij}$$

where ε is the spatial scale at which oscillations of vorticity occur and where c is a constant which is not, exactly known and depends on some mean decorrelation time of the system (see Appendix A).

Thus, assuming that for $\beta = 0$ the p_i are advected like passive scalars leads us to identify

$$A(\mathbf{x}) = c\varepsilon^2\, \text{Ln}\left(\frac{r}{\varepsilon}\right) (\overline{\omega^2} - \bar{\omega}^2) \tag{8}$$

To summarize, we have found that the evolution of the p_i is given by the following set of convection-diffusion equations:

$$(\text{RE}_n) \quad \begin{cases} (p_i)_t + \nabla \cdot (p_i \bar{\mathbf{u}} + \mathbf{J}_i) = 0, & i = 1,\dots, n \\ \mathbf{J}_i = -A(\mathbf{x})[\nabla p_i - \beta(\bar{\omega} - a_i)\, p_i\, \nabla\psi] \\ \mathbf{J}_i \cdot n = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where $A(\mathbf{x})$ is given by (8) and β is determined at each time by the conservation of the energy (7).

Remark. In ref. 27 we have shown, in the particular case of a vortex patch, that these equations can be obtained by using only the recipes of linear thermodynamics about the equilibrium. Unfortunately this method does not extend to the n-level case.

Remark. To get the formula (8) we have made the assumption that the oscillations of vorticity occur at a well-defined spatial scale ε (see Appendix A). This is of course a great simplification, since, on the one hand, these oscillations may concern a large range of scales and, on the other hand, such a scale ε would vary with space and time. It is well known that as the flow evolves the oscillations tend to reach smaller and smaller scales.

In practice, we shall consider $c\varepsilon^2 \operatorname{Log} r/\varepsilon$ as an empirical coefficient, the value of which will be fixed at our convenience for computational purposes (this will be discussed in Section 4). Notice that the variable coefficient $A(\mathbf{x})$ given by (8) has the dimension of a viscosity and vanishes where $\overline{\omega^2} - \bar{\omega}^2 = 0$, i.e., where there is no mixing of the vorticity at small scales.

Remark. In the particular case $\Omega = B(0, R)$ we must also take into account the constraint given by the angular momentum. Then the currents have to satisfy the supplementary condition $\int_\Omega \mathbf{x} \cdot \mathbf{J}_\omega \, d\mathbf{x} = 0$. Thus the optimal currents can be written

$$\mathbf{J}_i = -A(\mathbf{x})[\nabla p_i - (\bar{\omega} - a_i) \, p_i(\beta \, \nabla\psi + \gamma\mathbf{x})]$$

where β and γ are determined by the linear system

$$\beta \int_\Omega \theta^2 (\nabla\psi)^2 \, d\mathbf{x} + \gamma \int_\Omega \theta^2 \mathbf{x} \cdot \nabla\psi \, d\mathbf{x} = -\int_\Omega \nabla\psi \cdot \nabla\bar{\omega} \, d\mathbf{x}$$

$$\beta \int_\Omega \theta^2 \mathbf{x} \cdot \nabla\psi \, d\mathbf{x} + \gamma \int_\Omega \theta^2 \mathbf{x}^2 \, d\mathbf{x} = -\int_\Omega \mathbf{x} \cdot \nabla\bar{\omega} \, d\mathbf{x} \tag{9}$$

where $\theta^2 = \overline{\omega^2} - \bar{\omega}^2$. We shall suppose that θ^2 is not identical to 0 on Ω (some mixing has occurred). Then, using the Schwarz inequality, we easily see that the determinant of this system is > 0, and the solution for β and γ is therefore unique, except in the degenerate case of a solid-body rotation.

In what follows we shall consider equations (RE_n) with A given by

$$A(\mathbf{p})(\mathbf{x}) = D(\overline{\omega^2}(\mathbf{x}) - \bar{\omega}^2(\mathbf{x})) \tag{10}$$

The link between (RE_n) and the equilibrium theory is given by the following convergence result.

Proposition 3.1. Let us suppose that the solution $p_i(t, x)$, $i = 1,..., n$, of (RE_n) converges (in a strong enough sense), when t goes to infinity, toward a stationary state $p_i^*(x)$.

Let us assume also that there is some open connected subdomain A of Ω satisfying $\bar{A} \subset \Omega$ and

   (∗) $p_i^*(x) > 0$ and $A(\mathbf{p}^*)(x) > 0$ for all x in $\bar{A}$.

   (∗∗) $\mathbf{u}^* \cdot \mathbf{n} = 0$ and $\mathbf{J}_i^* \cdot \mathbf{n} = 0$ on ∂A [$\mathbf{n}$ is the unit vector normal to the boundary ∂A, $\mathbf{u}^*$ is the velocity field associated to $\bar{\omega}_* = \sum a_i p_i^*$, and $\mathbf{J}_i^*$ is the current associated to p_i^* by (6)].

Then $p_i^*(x)$ is a Gibbs state on A; that is, there are parameters $\alpha_1^*,..., \alpha_n^*$ ($\alpha_n^* = 0$) and β^* such that

$$p_i^*(x) = \frac{\exp(-\alpha_i^* - \beta^* a_i \psi^*(x))}{Z(\psi^*(x))} \qquad \text{for all} \quad x \text{ in } A$$

Moreover, $\beta^* = \lim_{t \to \infty} \beta(t)$, where $\beta(t)$ is given at each time by (7).

Proof. The proof mainly reproduces the arguments given in ref. 27.

When $t \to \infty$ we have $p_i(t, x) \to p_i^*(x)$, $\bar{\omega}(t, x) \to \bar{\omega}_*(x)$, $\psi(t, x) \to \psi^*(x)$, and $\beta(t) \to \beta^*$ [given by (7), where we replace $p_i(t, x)$ by $p_i^*(x)$]. We have also

$$\mathbf{J}_i(t) = -A(\mathbf{p}(t))[\nabla p_i(t) - \beta(t)(\bar{\omega}(t) - a_i) p_i(t) \nabla \psi] \to \mathbf{J}_i^*$$

and since the $p_i(t, x)$ satisfy (RE_n), the functions $p_i^*(x)$ will satisfy the set of stationary equations

$$\nabla \cdot (p_i^* \mathbf{u}^* + \mathbf{J}_i^*) = 0, \qquad i = 1,..., n$$

Now let us calculate

$$\int_A \sum_i \text{Ln } p_i^* \, \nabla \cdot (p_i^* \mathbf{u}^* + \mathbf{J}_i^*) \, dx$$

$$= \int_A \sum_i \text{Ln } p_i^* \, \nabla p_i^* \cdot \mathbf{u}^* \, dx + \int_A \sum_i \text{Ln } p_i^* \, \nabla \cdot \mathbf{J}_i^* \, dx$$

Integrating by parts, we see that the first term is zero (due to $\nabla \cdot \mathbf{u}^* = 0$), while the second gives

$$-\int_A \sum_i \frac{\nabla p_i^*}{p_i^*} \cdot \mathbf{J}_i^* \, d\mathbf{x} = -\int_A \sum_i \frac{1}{p_i^*} [\nabla p_i^* - \beta^*(\bar{\omega}_* - a_i) \, p_i^* \, \nabla\psi^*] \cdot \mathbf{J}_i^* \, d\mathbf{x}$$

$$-\beta^* \int_A \sum_i (\bar{\omega}_* - a_i) \, \nabla\psi^* \cdot \mathbf{J}_i^* \, d\mathbf{x}$$

but the last term vanishes since $\sum \mathbf{J}_i^* = 0$ and $\int_A \nabla\psi^* \cdot \mathbf{J}_i^* \, d\mathbf{x} = 0$ [this last equality comes from $\int_A \psi^* \nabla \cdot (p_i^* \mathbf{u}^* + \mathbf{J}_i^*) \, d\mathbf{x} = 0$, by integration by parts].
We finally get

$$\int_A \sum_i \frac{1}{p_i^*} [\nabla p_i^* - \beta^*(\bar{\omega}_* - a_i) \, p_i^* \, \nabla\psi^*]^2 \, A(\mathbf{p}^*) \, d\mathbf{x} = 0$$

from which

$$\nabla \mathrm{Ln} \, p_i^* - \beta^*(\bar{\omega}_* - a_i) \, \nabla\psi^* = 0 \quad \text{on } A \qquad \text{for} \quad i = 1,\dots,n$$

Subtracting equation n from equation i, we deduce that $\mathrm{Ln}(p_i^*/p_n^*) + \beta^*(a_i - a_n) \, \psi^*$ has a constant value $-\alpha_i$ on A. Now, using the relationship $\sum p_i^*(\mathbf{x}) = 1$, we deduce that the p_i^* satisfy the Gibbs state relationships (3) on A.

3.2.3. The General Case.

The above analysis extends to the general case of a continuous initial vorticity distribution π_0. Then the macroscopic description of the flow is given at time t and position $\mathbf{x}$ by the probability distribution of vorticity $\rho(t, \mathbf{x}, z) \, \pi_0$.

Applying the same method as for the n-level case, we get the following set of relaxation equations:

$$(\mathrm{RE}_\alpha) \quad \begin{cases} \rho_t + \nabla_x \cdot (\rho \bar{\mathbf{u}} + \mathbf{J}_z) = 0, & \forall z \\ \mathbf{J}_z = -A(\rho)[\nabla_x \rho - \beta(\bar{\omega} - z) \, \rho \, \nabla\psi] \\ \mathbf{J}_z \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

where the variable z appears as a parameter, $A(\rho)$ is still given by (10), $\bar{\omega}(t, \mathbf{x}) = \int z \rho(t, \mathbf{x}, z) \, d\pi_0(z)$, ψ is the corresponding stream function, and β is given by the conservation of energy (7).

Of course this system is not easily tractable. It is convenient to introduce for $k = 1, 2,\dots$ the momentum $\overline{\omega^k}(t, \mathbf{x}) = \int z^k \rho(t, \mathbf{x}, z) \, d\pi_0(z)$ and the

corresponding currents $\mathbf{J}_k = \int z^k \mathbf{J}_z \, d\pi_0(z)$. Then (RE_∞) straightforwardly gives the infinite hierarchy of equations:

$$(\mathrm{RM}_\infty) \quad \begin{cases} \partial_t \overline{\omega^k} + \nabla_x \cdot (\overline{\omega^k} \, \bar{\mathbf{u}} + \mathbf{J}_k) = 0, & k = 1, 2, \dots \\ \mathbf{J}_k = -A(\rho)[\nabla_x \overline{\omega^k} - \beta(\bar{\omega}\overline{\omega^k} - \overline{\omega^{k+1}}) \nabla \psi] \\ \mathbf{J}_k \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

3.3. Toward a Modeling of Real Turbulent Flows

Let us return to the case of a slightly viscous flow and suppose that the energy and the N functionals $\Gamma_1, \dots, \Gamma_N$ are conserved. Then it is natural to truncate the above system (RM_∞) and consider only the functions $\overline{\omega^1}, \dots, \overline{\omega^N}$. To get a closed set of equations, we need then to deduce $\overline{\omega^{N+1}}(\mathbf{x})$ from $\overline{\omega^1}(\mathbf{x}), \dots, \overline{\omega^N}(\mathbf{x})$. This can be done naturally by a maximum entropy procedure. At each point $\mathbf{x}$ let us denote now $\rho(\mathbf{x}, z) \pi_0$ the probability distribution which solves the variational problem (VP.2):

$$\int \rho \, \mathrm{Ln} \, \rho \, d\pi_0(z) = \min \int \theta \, \mathrm{Ln} \, \theta \, d\pi_0(z)$$

under the constraints

$$\int z^k \theta(z) \, d\pi_0(z) = \overline{\omega^k}(\mathbf{x}), \qquad k = 1, \dots, N$$

Now we define the closure relationship by

$$\overline{\omega^{N+1}}(\mathbf{x}) = \int z^{N+1} \rho(\mathbf{x}, z) \, d\pi_0(z)$$

This gives a well defined dynamical system (RM_N) for the functions $\overline{\omega^1}(t, \mathbf{x}), \dots, \overline{\omega^N}(t, \mathbf{x})$.

Notice that $\rho(t, \mathbf{x}, z) = (1/Z) \exp[-\sum \alpha_k(t, \mathbf{x}) z^k]$, where $\alpha_1, \dots, \alpha_N$ are the Lagrange multipliers of the above constraints, and that (RM_N) gives also the evolution of the Young measure $\rho(t, \mathbf{x}, z) \pi_0$.

As for (RE_n), the link with the equilibrium theory is confirmed by a convergence result.

Proposition 3.2. Let us suppose that the solution $\overline{\omega^1}(t, \mathbf{x}), \dots, \overline{\omega^N}(t, \mathbf{x})$ of (RM_N) converges, when t goes to infinity, toward some stationary state $\overline{\omega^1_*}, \dots, \overline{\omega^N_*}(\mathbf{x})$, and suppose, as in Proposition 3.1, that there is an open domain Λ such that:

($*$) For all $\mathbf{x}$ in $\bar{\Lambda}$, the unique solution of the variational problem (VP.2) is written $\rho^*(\mathbf{x}, z)\,\pi_0$, where $\rho^*(\mathbf{x}, z) = (1/Z)\exp[-\sum_k \alpha_k^*(\mathbf{x})\,z^k]$, $\alpha_k^*(\mathbf{x})$ being the Lagrange multiplier associated to the constraint $\int z^k \theta(z)\,d\pi_0(z) = \overline{\omega^k}(\mathbf{x})$, $k = 1,..., N$; and $A(\rho^*)(\mathbf{x}) > 0$ on $\bar{\Lambda}$.

($**$) $\overline{\omega^{N+1}}(t, \mathbf{x})$ converges toward $\overline{\omega_*^{N+1}}(\mathbf{x})$ on $\bar{\Lambda}$.

($***$) $\mathbf{u}^* \cdot \mathbf{n} = 0$ and $\mathbf{J}_k^* \cdot \mathbf{n} = 0$ (for $k = 1,..., N$) on $\partial\Lambda$.

Then $\rho^*(\mathbf{x}, z)$ satisfies the relationship (3') of Gibbs states on Λ.

Proof. As for Proposition 3.1, we get

$$\nabla \cdot (\overline{\omega_*^k}\mathbf{u}^* + \mathbf{J}_k^*) = 0, \qquad k = 1,..., N$$

Let us now calculate (integrate by parts)

$$\int_\Lambda \sum_k \alpha_k^* \nabla \cdot (\overline{\omega_*^k \mathbf{u}^*} + \mathbf{J}_k^*)\,d\mathbf{x} = \int_\Lambda \sum_k \nabla\alpha_k^* \cdot \overline{\omega_*^k}\mathbf{u}^*\,d\mathbf{x} + \int_\Lambda \sum_k \nabla\alpha_k^* \cdot \mathbf{J}_k^*\,d\mathbf{x}$$

but $Z(\mathbf{x}) = \int \exp[-\sum_k \alpha_k^*(\mathbf{x})\,z^k]\,d\pi_0(z)$, and a straightforward calculation yields $\nabla \operatorname{Ln} Z = -\sum_k \overline{\omega_*^k}\,\nabla\alpha_k^*$, so that the first integral vanishes.

Let us define $\mathbf{J}_z^* = -A(\rho^*)[\nabla_x \rho^* - \beta^*(\overline{\omega_*^1} - z)\,\rho^*\,\nabla\psi^*]$, so that $\mathbf{J}_k^* = \int z^k \mathbf{J}_z^*\,d\pi_0(z)$. From the equality $\nabla_x\rho^*/\rho^* = -\nabla \operatorname{Ln} Z - \sum_k \nabla\alpha_k^*$, we get

$$\int_\Lambda \sum_k \nabla\alpha_k^* \cdot \mathbf{J}_k^*\,d\mathbf{x} = -\int_\Lambda d\mathbf{x} \int \left(\frac{\nabla_x\rho^*}{\rho^*} + \nabla \operatorname{Log} Z\right) \cdot \mathbf{J}_z^*\,d\pi_0(z)$$

$$= -\int_\Lambda d\mathbf{x} \int \left[\frac{\nabla_x\rho^*}{\rho^*} - \beta^*(\bar{\omega}_* - z)\,\nabla\psi^*\right] \cdot \mathbf{J}_z^*\,d\pi_0(z)$$

since $\int \mathbf{J}_z^*\,d\pi_0(z) = 0$ and $\int_\Lambda \nabla\psi^* \cdot \mathbf{J}_1^*\,d\mathbf{x} = 0$. Finally, we get

$$\int_\Lambda d\mathbf{x} \int [\nabla_x\rho^* - \beta^*(\bar{\omega}_* - z)\,\rho^*\,\nabla\psi^*]^2\,\frac{A(\rho^*)}{\rho^*}\,d\pi_0(z) = 0$$

from which $\nabla_x \operatorname{Ln}\rho^* - \beta^*(\bar{\omega}_* - z)\,\nabla\psi^* = 0$, for $d\mathbf{x}$-almost every $\mathbf{x}$ in Λ and π_0-almost every z. That is,

$$\nabla \operatorname{Ln} Z + \sum_k z^k\,\nabla\alpha_k^* + \beta^*(\bar{\omega}_* - z)\,\nabla\psi^* = 0$$

Now, for any fixed $\mathbf{x}$ in Λ, we identify the polynomial in the variable z (π_0 is diffuse), which gives

$$\nabla \operatorname{Ln} Z + \beta^* \bar{\omega}_* \nabla \psi^* = 0$$

$$\nabla \alpha_1^* - \beta^* \nabla \psi^* = 0$$

$$\nabla \alpha_2^* = 0$$

$$\vdots$$

$$\nabla \alpha_N^* = 0$$

so that ρ^* satisfies the relationship of Gibbs states on Λ.

4. EQUATIONS (RE_n) AND LARGE-EDDY SIMULATIONS

This section is a first step toward a complete study of the relevance of equations (RE_n) and (RM_N).

Our purpose.here is to compare the results of numerical simulations using Navier–Stokes equations at large Reynolds number with those obtained from (RE_n) with a much larger viscosity coefficient.

4.1. Choice of the Initial State

We have chosen to simulate the formation of a tripolar coherent structure (in the whole plane). This phenomenon has been already studied experimentally[10, 32] and numerically.[3, 4, 22]

As usual, we can approximate the flow in the whole plane by taking an initial datum localized in a small portion of a (comparatively) large periodic domain $\Omega = \,]0, 2\pi[\,\times\,]0, 2\pi[$. Our initial vorticity function is

$$\omega_0(\mathbf{x}) = a_1 \quad \text{in the ellipse} \quad \frac{(x_1 - \pi)^2}{r_0^2} + \frac{(x_2 - \pi)^2}{r_1^2} \leqslant 1$$

$$\omega_0(\mathbf{x}) = a_2 \quad \text{in the annulus} \quad r_2^2 \leqslant (x_1 - \pi)^2 + (x_2 - \pi)^2 \leqslant r_3^2$$

$$\omega_0(\mathbf{x}) = 0 \quad \text{elsewhere}$$

Due to the periodicity of the flow in the two directions, the mean value of ω_0 on Ω must be zero, so that the parameters $a_1, a_2, r_0, r_1, r_2, r_3$ satisfy $a_1 r_0 r_1 = a_2(r_2^2 - r_3^2)$.

In what follows, we shall take

$$a_2 = 2\pi, \quad r_0 = 0.5, \quad r_1 = 0.3, \quad r_2 = 0.65, \quad r_3 = 1 \quad (\text{or } 0.8)$$

so that ω_0 is an elliptical patch of positive vorticity surrounded by an annulus of negative vorticity.

Let us notice that our initial datum is not axially symmetric (compare to refs. 3, 4, and 22). The advantage of this is that we do not need to destabilize the system: the mixing process occurs instantly and the final structure is reached more rapidly.

4.2. Navier–Stokes Simulations

We solve numerically, in a classical way, Navier–Stokes equations with periodic boundary conditions. The spatial derivatives are treated by a pseudo-spectral method[11] and the time discretization scheme is a third-order Adams–Bashforth scheme.

4.2.1. Numerical Parameters. (i) Spatial resolution: the highest spatial resolution is of course desirable in order to reach high Reynolds number and properly handle the initial vorticity discontinuity. We chose 256 gridpoints in each direction, which allows us to approach the inertial limit with a reasonable computing time.

(ii) Reynolds number: Since the typical velocity and length are unity, $Re = 1/v(v = \text{viscosity})$ is the actual Reynolds number: we take $Re = 2000$, which is the highest Reynolds number compatible with our resolution.

(iii) Time step: $\Delta t = 0.001$.

4.2.2. The Results. For $r_3 = 1$ the simulated flow is described in Fig. 1a: the evolution of the flow is represented by successive snapshots of the vorticity field at different times. We see the intricate motion of the fluid yielding the mixing of the vorticity levels $0, a_1, a_2$. After this mixing process the system stabilizes into a final tripolar vortex structure which only slowly diffuses by viscosity.

In the final state the system has a steady configuration in a rotating reference frame. This can be checked on the scatterplot of the vorticity versus stream function in this frame (which is determined "experimentally") shown in Fig. 5a. The (ω, ψ) relationship can be analyzed as the superposition of two curves; this indicates that the flow is a stationary solution of the Euler equations. This graph is typical of the final state reached by a three-level system (see also refs. 4 and 22); it indicates that the system has not reached a global statistical equilibrium in the whole plane, but only in a bounded region. The branch containing the zero-vorticity level disappears if we restrict consideration to the tripolar structure where the $\omega = f(\psi)$ relationship predicted by the statistical equilibrium theory only remains.
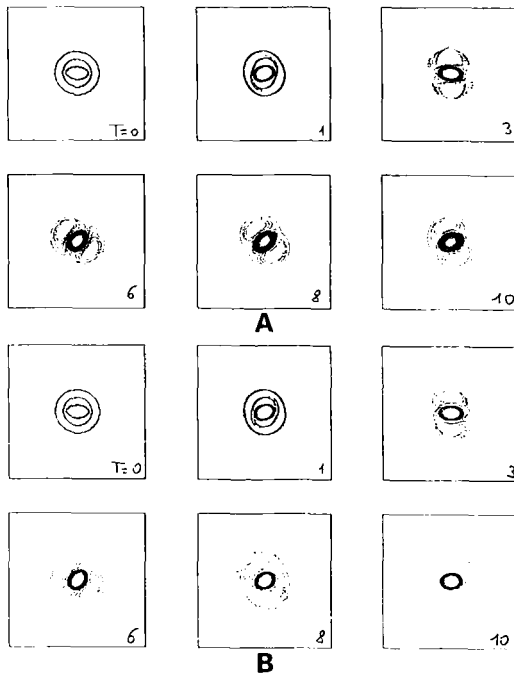
Fig. l. Successive snapshots of the vorticity field ($r_3 = 1$.). The contour interval is 2.4. (a) Navier–Stokes, Re = 2000. (b) (RE₃), variable viscosity $v = 10^{-2}$.

During the process the overall relative loss of energy is about 6%, while for the angular momentum it is 10^{-3}, which indicates that the approximation by a periodic box is accurate.

For $r_3 = 0.8$ the evolution of the flow, which still exhibits a strong mixing of vorticity at small scales, is different at large scale, where we observe a splitting into two dipolar coherent structures (see Fig. 4a).

4.3. (RE₃) Computations

4.3.1. Constant Viscosity: $A(x) = v$. In this case, equations (RE₃) are very similar to Navier–Stokes equations (with viscosity v) in the (ω, ψ) formulation. Therefore we can apply to (RE₃) the same numerical treatment (described above). The main change is that we need to compute at each time the multipliers β, γ by solving the system (9).

We see in Fig. 3 the result obtained for $r_3 = 1$ and $v = 10^{-2}$ (the flow is represented by the mean vorticity field). We observe the transient appearance of a tripolar structure; it disappears as time evolves, while the vorticity diffuses indefinitely in the surrounding space.

4.3.2. Variable Viscosity.   We take

$$A(\mathbf{x}) = \nu \frac{\sum a_i^2 p_i - (\sum a_i p_i)^2}{\text{Norm}}$$

where the normalization factor

$$\text{Norm} = \frac{a_1^2 + a_2^2}{2} - \left(\frac{a_1 + a_2}{2}\right)^2$$

We may think *a priori* that a viscosity which can vanish at some places will cause numerical difficulties, introducing spurious oscillations at small scales. In fact this does not occur here: our variable viscosity coefficient vanishes only where the flow has no tendency to develop small-scale oscillations, whereas where oscillations would occur the factor $\sum a_i^2 p_i - (\sum a_i p_i)^2$ is positive and they are smoothed out.
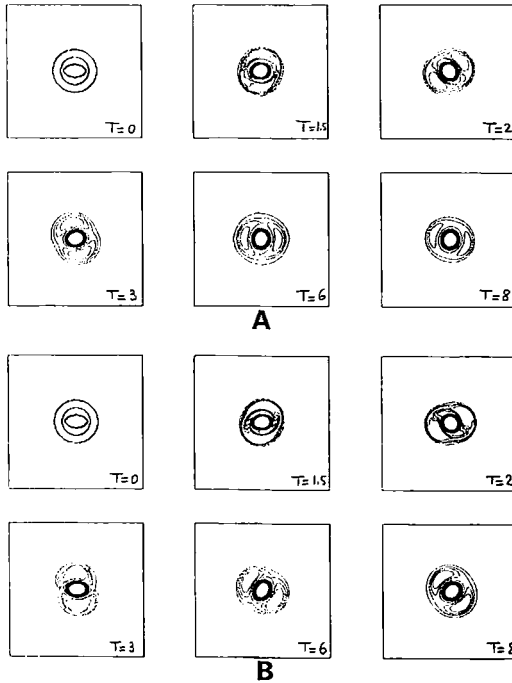

Fig. 2.   Successive snapshots of the vorticity field $(r_3 = 1.)$. The contour interval is 2.4.
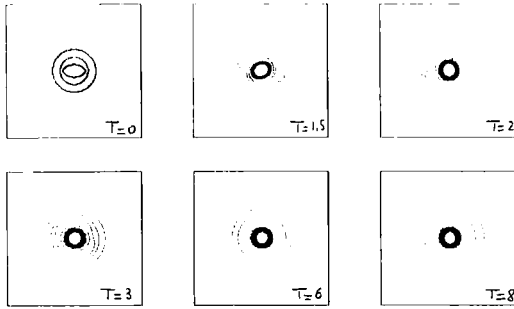(a) (RE_3), variable viscosity $\nu = 10^{-1}$. (b) (RE_3), variable viscosity $\nu = 2 \times 10^{-2}$.

Fig. 3.   Successive snapshots of the vorticity field ($r_3 = 1.$). The contour interval is 2.4. (RE_3), constant viscosity $v = 10^{-2}$.
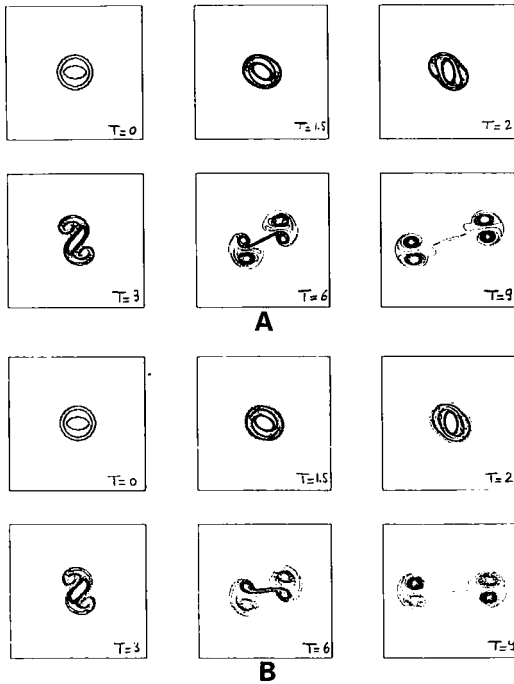

Fig. 4.   Successive snapshots of the vorticity field ($r_3 = 0.8$). The contour interval is 2.4. (a) Navier–Stokes, $Re = 2000$. (b) (RE_3), variable viscosity $v = 10^{-2}$.
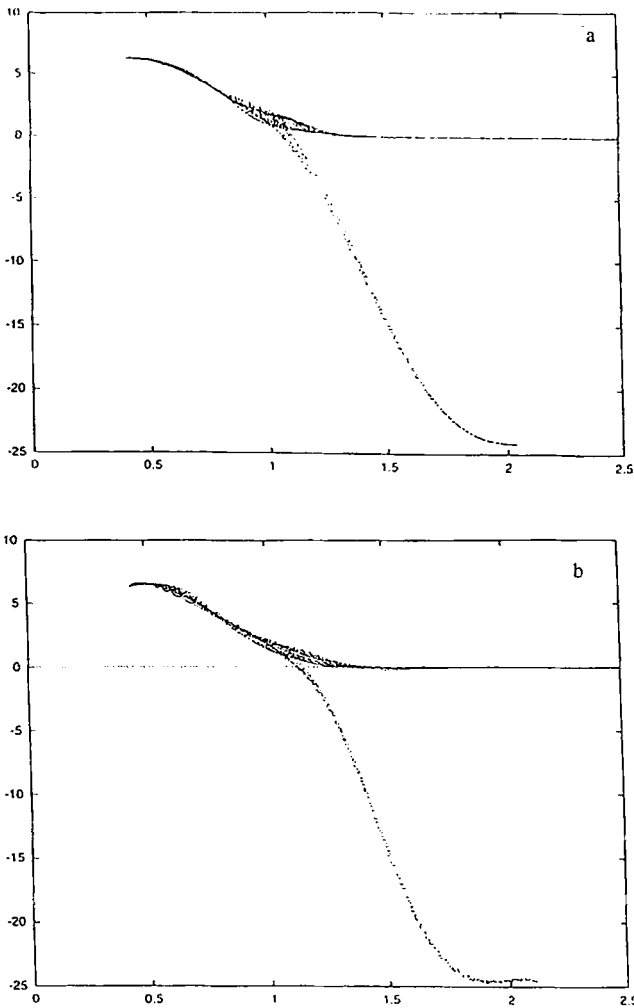
Fig. 5.   Scatterplot of vorticity versus stream function at $T = 8$. (a) Navier–Stokes, Re $= 2000$ (cf. Fig. 1a). (b) (RE_3), variable viscosity $v = 10^{-2}$ (cf. Fig. 1b). (c) (RE_3), variable viscosity $v = 10^{-1}$ (cf. Fig. 2a).

   The numerical method used for Navier–Stokes equations and (RE_3) with a constant viscosity works fairly well in that case also. All the other numerical parameters being unchanged, we present in Figs. 1b, 2a, and 2b the results for $r_3 = 1$ and different values of $v(v = 10^{-2}, 10^{-1}, 2 \times 10^{-2})$.
   Whereas the results for $v = 10^{-2}$ and 2×10^{-2} are quite indistinguishable and very close to those obtained by Navier–Stokes simulations (see
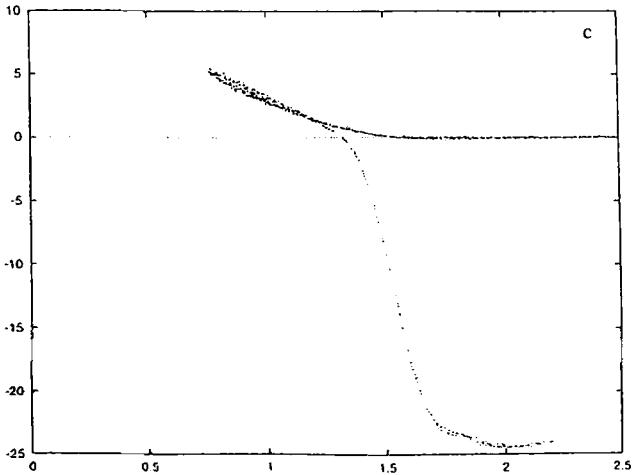
Fig. 5   (*continued*)

Figs. 1 and 2b); for $\nu = 10^{-1}$, the large-scale motion is noticeably different: the formation of the coherent structure is speeded up. This difference persists in the final state, as shown in Fig. 5c.

The case $\nu = 10^{-2}$ and $r_3 = 0.8$, displayed in Fig. 4, also shows very good agreement with the Navier–Stokes simulations in a more complex case where the flow does not converge rapidly toward an equilibrium state.

Remark.   From the formula (7) giving β we see that for an initial datum which is not mixed (i.e., $\overline{\omega^2} - \bar{\omega}^2 = 0$ everywhere) β is not defined at $t = 0$: the dynamical system is not defined before some diffusion occurs. To overcome this difficulty, one can proceed as follows. Take at the beginning the modified expression for the currents $\mathbf{J}_i = -A[\nabla p_i + \beta' a_i \nabla \psi]$, where β' is determined by the conservation of the energy: $\int_\Omega \nabla \psi \cdot \mathbf{J}_\omega \, d\mathbf{x} = 0$; solve this modified equation until some diffusion occurs and then come back to (RE_n).

4.4. Conclusion

The mathematical properties of (RE_n) such as the existence and uniqueness of the solution will be studied elsewhere. Of course we expect that when $\nu \to 0$ the mean vorticity associated with the solution of (RE_n) converges toward the solution of the Euler equations.

The main conclusion of our numerical tests is that the large-scale motion is not very sensitive to the value of ν. We actually observe that beyond some value (about 10^{-2} in our example), the large-scale motion no

longer depends on v. The important fact is that this value is large compared to the viscosity needed to reproduce the inertial organization of the large-scale flow by Navier–Stokes computations. The consequence is that we can reproduce precisely the large-scale flow by using model equations with large diffusive effects. From a practical standpoint it gives us the opportunity to perform large-eddy simulations at a much lower computational cost than by (high-resolution) Navier–Stokes simulations.[9]

5. COMMENTS

We have proposed in this paper a method which exploits the statistical equilibrium theory to construct new evolution equations. These equations intend to model the effect of the small scales on the large-scale motion in 2D turbulent flows.

In our approach we used a variational principles the maximum entropy production principle, to overcome our poor knowledge of the detained dynamical behavior of the Euler equations. This principle, combined with dynamical considerations on the diffusion currents, yields a convenient explicit expression for the turbulent-diffusion terms. Of course, we can only postulate the validity of this approach. This principle, which can be viewed as a pragmatic way to deal with our ignorance, seems of general scope; but its range of applications is not clearly identified and it has to be considered with much care.

Our equations (RE_n) or (RM_N) are of a convection-diffusion type, with turbulent diffusion terms which can vanish. More precisely, the turbulent diffusion is active where the fluid motion develops small-scale vorticity oscillations, then it smoothes them out and makes the entropy increase. We see that this process is favorable from a computational standpoint.

In practice, the choice of the empirical viscosity v results from an adjustment: with low values of v the vorticity structures at the grid scale (supposed given) are not sufficiently damped and numerical errors appear, whereas with large values the main features of the large-scale dynamics can be altered (even if we can predict satisfactorily the final state of the system). In fact, we observed that there is a large range of values of v giving, with a good approximation, the same large-scale motion. The consequence is that we can mimic the flow obtained from high-Reynolds-number Navier–Stokes direct numerical simulations by using equations with large diffusive effects. Decreasing the value of v needs to increase the spatial resolution accordingly. This seems to have only little effect on the large-scale motion, while it displays further details of the flow at lower spatial scales.

The numerical simulations of Section 4 concern only equations (RE_n) for $n = 3$. The formation of a tripolar coherent structure was recently studied both numerically[10, 32] and experimentally.[3, 4, 22] We have intentionally considered this example because it is rather subtle: there is no maximum entropy state in the whole space corresponding to our initial state, so that the final structure results both from the tendency of the system to maximize the entropy functional and from the dynamics, which tends to freeze the system into some local Gibbs state.

We observe through this example a striking agreement between the numerical simulations using (RE_n) and the results from high-Reynolds-number Navier–Stokes direct simulations.

By the way, this numerical study also provides a relevant test for the equilibrium theory for a three-level system, since the relationship $\bar{\omega} = f(\psi)$ in the final state is in very good agreement with the Gibbs-state relationship predicted by our model (this relationship closely depends on the particular form of our entropy functional).

Up to now, wee have only considered the case $\varepsilon = \text{const}$. This is a crude approximation since the scale $\varepsilon(t, \mathbf{x})$ of the vorticity oscillations certainly varies with space and time; moreover, $\varepsilon(t, \mathbf{x}) \to 0$ when $t \to \infty$, due to the straining by the mean flow. We can take this into account to get more sophisticated models.

The study of equations (RM_N) is in progress. They are better suited to handle more general situations, since we are no longer constrained to work with vorticity functions taking only a finite number of values.

As noticed in Section 4, our models can describe accurately the large-scale motion without having to handle explicitly the small scales. This is due to the fact that the small-scale "chaos" produced by the intricate dynamics of the Euler equations is efficiently taken into account by statistical mechanics.

But why does statistical mechanics work? This is certainly due to some kind of ergodicity of the perfect fluid dynamics. This issue was clearly addressed by Arnold,[1] who showed that the perfect fluid motion is a geodesic flow on a Riemannian manifold whose sectional curvature is negative (for most of the sections). This property which implies the exponential instability of the flow, is nothing but the celebrated "butterfly effect." It is this "chaotic" small-scale behavior of the flow which actually make the large-scale organization predictible (at low computational cost) by statistical mechanics devices.

In a recent paper Constantin and Wu[35] discuss the inviscid limit of solutions of the Navier–Stokes equations in the case of a nonsmooth initial vorticity. They find that the most accurate companion to a solution of the Navier–Stokes equations might be a mollified Euler solution.

More precisely, let $\omega^{NS}(t, x)$ and $\omega(t, x)$ be (respectively) the solutions of the Navier–Stokes (with viscosity ν) and Euler equations corresponding to the same initial data $\omega_0(x)$. They proved the estimate

$$\|\omega^{NS}(t,...) - \omega_\delta(t,...)\|_{L^p} \leqslant K_p(\omega_0, T)\, \nu^{s/2p} \qquad \text{for} \quad 0 \leqslant t \leqslant T$$

where $\delta = (\nu/\|\omega_0\|_\infty)^{1/2}$, $\omega_\delta(x) = \int \phi_\delta(y)\, \omega(x - y)\, dy$, $\phi_\delta(y) = \delta^{-2}\phi(y/\delta)$, and ϕ is any mollifier. ω_δ obviously satisfies the following equation:

$$(E_\delta) \qquad \frac{\partial}{\partial t}\, \omega_\delta + \operatorname{div}(\mathbf{u}_\delta \omega_\delta) + \operatorname{div}(\mathbf{J}_\delta) = 0$$

where $\mathbf{J}_\delta = (\mathbf{u}\omega)_\delta - \mathbf{u}_\delta \omega_\delta$.

Of course (E_δ) is not closed; and since ω is not regular and generally has small-scale oscillations, we do not know how to get closure relationships (to express $\mathbf{J}_\delta$ in terms of ω_δ).

Let us now try to make a link with our approach (at least in the case where ω_0 is a vortex patch). Since the statistical equilibrium theory seems to predict accurately the coherent states of the inviscid system, we expect that (E_δ) increases the entropy of the system, while the macroscopic state ω_δ converges toward the equilibrium. In this context, our appeal to the MEPP to set up the relaxation equations appears as a convenient recipe to close equation (E_δ).

APPENDIX A. DIFFUSION OF A PASSIVE SCALAR

A1. General Framework

We consider the situation where a scalar density ρ is convected by an incompressible velocity field affected by small random fluctuations:

$$\mathbf{u}(t, \mathbf{x}, \zeta) = \bar{\mathbf{u}}(t, \mathbf{x}) + \tilde{\mathbf{u}}(t, \mathbf{x}, \zeta)$$

where $\mathbf{x}$ stays in a bounded regular domain Ω of $\mathbb{R}^d$, $\mathbf{u}(t, \mathbf{x}, \zeta) \in \mathbb{R}^d$, and ζ is some random parameter. We shall assume that $\bar{\mathbf{u}} \gg \tilde{\mathbf{u}}$ and that the mean value $\langle \tilde{\mathbf{u}}(t, \mathbf{x}, \zeta) \rangle = 0$ for all t, $\mathbf{x}$. We shall suppose also that $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$ ($\mathbf{n}$ is the normal unit vector at the boundary of Ω), and $\nabla \cdot \mathbf{u} = 0$, so that $\nabla \cdot \bar{\mathbf{u}} = 0$ and $\nabla \cdot \tilde{\mathbf{u}} = 0$.

We assume that the mean flow $\bar{\mathbf{u}}(t, \mathbf{x})$ is regular. with Lipschitz constant K:

$$|\bar{\mathbf{u}}(t, \mathbf{x}) - \bar{\mathbf{u}}(t, \mathbf{x}')| \leqslant K\, |\mathbf{x} - \mathbf{x}'|$$

We denote $\rho(t, \mathbf{x}, \zeta)$ the solution of the linear transport equation:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) \end{cases}$$

We want to get the equation satisfied by $\bar{\rho}(t, \mathbf{x}) = \langle \rho(t, \mathbf{x}, \zeta) \rangle$.

This is a classical problem and it is well known that in good cases $\bar{\rho}$ will satisfy a convection-diffusion equation. It can be established by rigorous mathematical methods in the case where $\tilde{\mathbf{u}}$ is a white noise process (see, for example, Freidlin and Wentzell[34]). Besides this idealized case, one needs to introduce additional assumptions on the random fluctuation $\tilde{\mathbf{u}}$ in order to show that $\bar{\rho}$ approximately satisfies an evolution equation of convection-diffusion type (see Kraichnan;[14] see also refs. 13, 15, and 20).

We shall make here the following assumptions:

There is some time $\tau > 0$, which is small with respect to the convective time scale $1/K$, such that:

(H1) For $|h| \leqslant \tau$, the following approximation holds:

$$\tilde{\mathbf{u}}(t + h, X(t + h, t, \mathbf{x}, \zeta), \zeta) \cong \tilde{\mathbf{u}}(t + h, \bar{X}(t + h, t, \mathbf{x}), \zeta)$$

(H2) Decorrelation hypothesis: For $|t - \theta| > \tau$, we have

$$\langle \tilde{\mathbf{u}}_j(\theta, \bar{X}(\theta, t, \mathbf{x}), \zeta) \, \tilde{\mathbf{u}}_i(t, \mathbf{x}, \zeta) \rangle = 0 \qquad \text{for all} \quad i, j$$

where the Lagrangian flows $X(t, s, \mathbf{x}, \zeta)$ and $\bar{X}(t, s, \mathbf{x})$ are defined by

$$\frac{\partial X}{\partial t} = \mathbf{u}(t, X, \zeta), \qquad X(s, s, \mathbf{x}, \zeta) = \mathbf{x}$$

$$\frac{\partial \bar{X}}{\partial t} = \bar{\mathbf{u}}(t, \bar{X}), \qquad \bar{X}(s, s, \mathbf{x}) = \mathbf{x}$$

With the assumptions H1 and H2 we can show by using classical arguments (we do not reproduce them here, for the sake of brevity) that $\bar{\rho}$ approximately satisfies the following equation of convection-diffusion type:

$$\bar{\rho}_t + \nabla \bar{\rho} \cdot \bar{\mathbf{u}} + \partial_i(D_{ij} \partial_j \bar{\rho}) = 0$$

where $\partial_i = \partial/\partial x_i$ and

$$D_{ij}(t, x) = -\frac{1}{\tau} \int_0^\tau dh \int_0^h \langle \tilde{u}_j(t + \theta, \bar{X}(t + \theta, t, \mathbf{x}), \zeta)$$

$$\times \tilde{u}_i(t + h, \bar{X}(t + h, t, \mathbf{x}), \zeta) \rangle \, d\theta \qquad \text{(A1)}$$

A2. From Vorticity Fluctuations to Velocity Fluctuations

Let us now focus on the case $d = 2$, where velocity fluctuations are caused by vorticity fluctuations. Let us consider a randomly fluctuating vorticity field $\omega(\mathbf{x}, \zeta)$; the corresponding velocity field $\mathbf{u}(\mathbf{x}, \zeta)$ is given by

$$
\begin{cases}
(\nabla \times \mathbf{u}) \cdot \mathbf{e}_3 = \omega \\
\nabla \cdot \mathbf{u} = 0 \\
\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial \Omega
\end{cases}
$$

that is, $\mathbf{u}(\mathbf{x}, \zeta) = \int_\Omega G(\mathbf{x}, \mathbf{x}') \, \omega(\mathbf{x}', \zeta) \, d\mathbf{x}'$, where $G(\mathbf{x}, \mathbf{x}')$ is the appropriate kernel.

We introduce now a simple stochastic model of the vorticity fluctuations. Let us consider two numbers ε, r such that $0 < \varepsilon \ll r \ll |\Omega|^{1/2}$, r is some fixed macroscopic scale, while the microscopic length scale ε is intended to go to zero. Let Ω_k be any equipartition of Ω (that is a partition of the set Ω into a finite number of disjoint subsets with the same area) such that $|\Omega|_k = \varepsilon^2$ and diameter$(\Omega_k) \leqslant \text{const} \cdot \varepsilon$, the constant being independent of the equipartition, and now define the random vorticity field

$$
\omega(\mathbf{x}, \zeta) = \sum_k \mathbf{1}_{\Omega_k}(\mathbf{x}) \, \omega_k(\zeta)
$$

where $\mathbf{1}_{\Omega_k}$ denotes the characteristic function of the set Ω_k and $\omega_k(\zeta)$ are independent (real-valued) random variables with mean values $\bar{\omega}_k$.

We shall write

$$
\bar{\omega}(\mathbf{x}) = \langle \omega(\mathbf{x}, \zeta) \rangle
$$

$$
\omega(\mathbf{x}, \zeta) = \bar{\omega}(\mathbf{x}) + \tilde{\omega}(\mathbf{x}, \zeta)
$$

The velocity fluctuation is thus given by

$$
\tilde{\mathbf{u}}(\mathbf{x}, \zeta) = \int_\Omega G(\mathbf{x}, \mathbf{x}') \, \tilde{\omega}(\mathbf{x}', \zeta) \, d\mathbf{x}'
$$

Let us now fix a point $\mathbf{x}$ in Ω, denote B_r the closed ball of radius r centered at $\mathbf{x}$, included in Ω, and write $G(\mathbf{x}, \mathbf{x}') = S(\mathbf{x}, \mathbf{x}') + R(\mathbf{x}, \mathbf{x}')$, where

$$
S(\mathbf{x}, \mathbf{x}') = \mathbf{e}_3 \times \frac{\mathbf{x} - \mathbf{x}'}{2\pi \, |\mathbf{x} - \mathbf{x}'|^2}
$$

is the singular part of the kernel and $R(\mathbf{x}, \mathbf{x}')$ is a smooth function at $\mathbf{x}' = \mathbf{x}$.

Let us define $\tilde{\mathbf{u}}_r(\mathbf{x}, \zeta) = \int_{B_r} S(\mathbf{x}, \mathbf{x}') \, \tilde{\omega}(\mathbf{x}', \zeta) \, d\mathbf{x}'$; then, writing $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_r + \tilde{\mathbf{v}}_r$, we get the following result:

Proposition A.1. Letting $\varepsilon \to 0$, we have $\langle \tilde{\mathbf{v}}_r(\mathbf{x}, \zeta)^2 \rangle = O(\varepsilon^2)$ and $\varepsilon^{-2} \langle \tilde{\mathbf{u}}_r(\mathbf{x}, \zeta)^2 \rangle \to +\infty$.

Proof.

First point. Let us define $\tilde{\mathbf{u}}_r^g(\mathbf{x}, \zeta) = \int_{B_r} G(\mathbf{x}, \mathbf{x}') \, \tilde{\omega}(\mathbf{x}', \zeta) \, d\mathbf{x}'$; one easily sees (use Schwarz's inequality) that

$$\langle (\tilde{\mathbf{u}}_r - \tilde{\mathbf{u}}_r^g)^2 \rangle \leqslant \varepsilon^2 \max_k \langle \tilde{\omega}_k^2 \rangle \int_{\Omega - Br} G(\mathbf{x}, \mathbf{x}')^2 \, d\mathbf{x}'$$

Similar calculations give

$$\langle (\tilde{\mathbf{u}}_r - \tilde{\mathbf{u}}_r^g)^2 \rangle \leqslant \varepsilon^2 \max_k \langle \tilde{\omega}_k^2 \rangle \int_{Br} R(\mathbf{x}, \mathbf{x}')^2 \, d\mathbf{x}'$$

and thus $\langle (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_r)^2 \rangle \leqslant C\varepsilon^2$.

Second point. We have

$$\tilde{\mathbf{u}}_r(\mathbf{x}, \zeta) = \sum_{\Omega_k \subset B_r} \tilde{\omega}_k(\zeta) \int_{\Omega_k} S(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}'$$

from where

$$\langle \tilde{\mathbf{u}}_r^2 \rangle = \sum_{\Omega_k \subset B_r} \langle \tilde{\omega}_k^2 \rangle \left(\int_{\Omega_k} S(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}' \right)^2$$

We shall suppose that $\langle \tilde{\omega}_k^2 \rangle = \langle \omega_k^2 \rangle - \bar{\omega}_k^2$ is approximately constant for $\Omega_k \subset B_r$ and equal to $\langle \omega(\mathbf{x}, \zeta)^2 \rangle - \bar{\omega}(\mathbf{x})^2$, so that

$$\langle \tilde{\mathbf{u}}_r^2 \rangle = \varepsilon^2 (\langle \omega^2 \rangle - \bar{\omega}^2) \sum_{\Omega_k \subset B_r} \frac{1}{|\Omega_k|} \left(\int_{\Omega_k} S(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}' \right)^2$$

and the result straightforwardly follows from the following lemma (whose proof is an easy exercise).

Lemma A.2. Let f be an integrable function on Ω, which is not square-integrable. Let $\mathcal{O} = \{\Omega_k\}$ be any equipartition of Ω. Then

$$\sum_k \frac{1}{|\Omega_k|} \left(\int_{\Omega_k} f(\mathbf{x}) \, d\mathbf{x} \right)^2 \to +\infty \qquad \text{when} \quad d(\mathcal{O}) \to 0$$

where we denote $d(\mathcal{O}) = \max_k$ diameter (Ω_k).

Let us now calculate the dominant part of the matrix

$$m_{ij}(\mathbf{x}) = \langle \tilde{u}_i(\mathbf{x}, \zeta) \, \tilde{u}_j(\mathbf{x}, \zeta) \rangle$$

We have

$$m_{ij}(\mathbf{x}) \underset{(\varepsilon \to 0)}{\approx} \langle \tilde{u}_{ri}(\mathbf{x}, \zeta) \, \tilde{u}_{rj}(\mathbf{x}, \zeta) \rangle$$

$$\approx (\langle \omega^2 \rangle - \bar{\omega}^2) \sum_{\Omega_k \subset B_r} \int_{\Omega_k} S_i(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}' \int_{\Omega_k} S_j(\mathbf{x}, \mathbf{x}') \, d\mathbf{x}'$$

To proceed further in the calculations, we need a combinatorial result (which is an easy consequence of a similar result: Lemma 4.1 given in ref. 16).

Lemma A.3. Let $\mathcal{O} = \{\Omega_l\}$ be an equipartition of Ω such that $|\Omega_l| = \varepsilon^2$ and $d(\mathcal{O}) \leqslant c\varepsilon$. Now for $k = (k_1, k_2) \in \mathbb{Z}^2$ let us define the points $\mathbf{x}^k = (\varepsilon(k_1 + 1/2), \varepsilon(k_2 + 1/2))$. Then $\mathcal{O}$ can be renumbered by the indices k such that $\mathbf{x}^k \in \Omega$ in such a way that

$$\delta(\mathbf{x}^k, \Omega_k) = \max_{x' \in \Omega_k} |\mathbf{x}^k - \mathbf{x}'| \leqslant (c + 1)\,\varepsilon$$

We shall suppose without loss of generality that $\mathbf{x} = 0$. Let us write

$$\int_{\Omega_k} S_i(0, \mathbf{x}') \, d\mathbf{x}' = |\Omega_k| \, S_i(0, \mathbf{x}^k) + R_{ik}$$

using the inequality

$$\left| \frac{\mathbf{x}'}{|\mathbf{x}'|^2} - \frac{\mathbf{x}^k}{|\mathbf{x}^k|^2} \right| \leqslant \frac{|\mathbf{x}' - \mathbf{x}^k|}{|\mathbf{x}'| \, |\mathbf{x}^k|}$$

we get

$$|R_{ik}| \leqslant \frac{\delta(\mathbf{x}^k, \Omega_k)}{2\pi \, |\mathbf{x}^k|} \int_{\Omega_k} \frac{1}{|\mathbf{x}'|} \, d\mathbf{x}'$$

Let us denote $\mathbf{x}^k = \varepsilon \mathbf{z}^k$ and $R_{ik} = \varepsilon r_{ik}$; then, up to some boundary terms (with negligible contribution), we have

$$\sum_{\Omega_k \subset B_r} \int_{\Omega_k} S_i(0, \mathbf{x}') \, d\mathbf{x}' \int_{\Omega_k} S_j(0, \mathbf{x}') \, d\mathbf{x}'$$

$$= \varepsilon^2 \sum_{|z^k| \leqslant r/\varepsilon} [\, S_i(0, \mathbf{z}^k) \, S_j(0, \mathbf{z}^k) + S_i(0, \mathbf{z}^k) \, r_{jk} + S_j(0, \mathbf{z}^k) \, r_{ik} + r_{ik} r_{jk} \,]$$

Now, one easily checks that the series $\sum S_i(0, \mathbf{z}^k) \, r_{jk}$, $\sum S_j(0, \mathbf{z}^k) \, r_{ik}$, and $\sum r_{ik} r_{jk}$ are summable for $k \in \mathbb{Z}^2$. Moreover,

$$\sum_{|\mathbf{z}^k| \leqslant r/\varepsilon} S_i(0, \mathbf{z}^k) \, S_j(0, \mathbf{z}^k) = 0 \qquad \text{for} \quad i \neq j$$

and $\sum_k S_i(0, \mathbf{z}^k)^2 = +\infty$, so that the dominant part of the matrix m_{ij} is

$$\varepsilon^2 \sum_{|\mathbf{z}^k| \leqslant r/\varepsilon} S_i(0, \mathbf{z}^k)^2$$

Easy calculations give

$$\sum_{|\mathbf{z}^k| \leqslant r/\varepsilon} S_i(0, \mathbf{z}^k)^2 \underset{(\varepsilon \to 0)}{\approx} \frac{1}{4\pi} \operatorname{Ln} \frac{r}{\varepsilon}$$

so that we finally get

$$m_{ij}(\mathbf{x}) \underset{(\varepsilon \to 0)}{\approx} (\langle \omega^2 \rangle(\mathbf{x}) - \bar{\omega}^2(\mathbf{x})) \frac{\varepsilon^2}{4\pi} \operatorname{Ln} \frac{r}{\varepsilon} \delta_{ij}$$

A3. Diffusion of a Passive Scalar

In our model we start with $\omega_0(\mathbf{x}, \zeta) = \sum \mathbf{1}_{\Omega_k}(\mathbf{x}) \, \omega_k(\zeta)$, and $\omega(t, \mathbf{x}, \zeta)$ is given by the flow of Euler equations; $\mathbf{u}(t, \mathbf{x}, \zeta)$ is the corresponding velocity field.

Let us calculate now the diffusion of a density ρ which is convected by $\mathbf{u}$. In order to apply the result of Section 1, we have to check that the assumptions H1, H2 are satisfied.

H1 is a consequence of the following estimate, whose proof is straightforward.

Lemma A.4. Let us suppose that $|\tilde{\mathbf{u}}(t, \mathbf{x}, \zeta)| \leqslant b$ (for all $t, \mathbf{x}, \zeta$) and that $|\bar{\mathbf{u}}(t, \mathbf{x}) - \bar{\mathbf{u}}(t, \mathbf{x}')| \leqslant K |\mathbf{x} - \mathbf{x}'|$ (K independent of t). Then, if we write $X(t, 0, \mathbf{x}, \zeta) = \bar{X}(t, 0, \mathbf{x}) + \tilde{X}(t, 0, \mathbf{x}, \zeta)$, we have

$$|\tilde{X}(t, 0, \mathbf{x}, \zeta)| \leqslant \frac{b}{K} (e^{Kt} - 1)$$

H1 then follows from the classical quasi-Lipschitz estimate:[33]

$$|\tilde{\mathbf{u}}(t, \mathbf{x}, \zeta) - \tilde{\mathbf{u}}(t, \mathbf{x}', \zeta)| \leqslant c(\Omega) \, |\omega|_\infty \, |\mathbf{x} - \mathbf{x}'| \, (1 + |\operatorname{Ln}| \, \mathbf{x} - \mathbf{x}'||)$$

Let us now consider the decorrelation hypothesis H2. It is rather natural to assume that there is some time τ_0 (decorrelation time) such that for $|t - \theta| > \tau_0$ the fields $\tilde{\omega}(\theta, \mathbf{x}, \zeta)$ and $\tilde{\omega}(t, \mathbf{x}, \zeta)$ are decorrelated, i.e.,

$$\langle \tilde{\omega}(\theta, \mathbf{x}, \zeta)\, \tilde{\omega}(t, \mathbf{x}', \zeta) \rangle = 0 \qquad \text{for all} \quad \mathbf{x}, \mathbf{x}'$$

Since

$$\tilde{\mathbf{u}}(t, \mathbf{x}, \zeta) = \int_\Omega G(\mathbf{x}, \mathbf{x}')\, \tilde{\omega}(t, \mathbf{x}', \zeta)\, d\mathbf{x}'$$

we shall have also

$$\langle \tilde{u}_i(t, \mathbf{x}, \zeta)\, \tilde{u}_j(t, \mathbf{x}, \zeta) \rangle = 0 \qquad \text{for} \quad |t - \theta| > \tau_0$$

This of course implies H2 if we make the additional strong assumption that $\tau_0 = \tau$.

Now we can apply the result of Section 1, and we get for the mean density a convection-diffusion equation with diffusion matrix $D_{ij}(t, \mathbf{x})$ given by (11).

Another consequence of Lemma A.4 is that $\tilde{\omega}(t, \mathbf{x}, \zeta)$ is approximately convected by the mean flow during the time interval $[0, \tau]$, i.e.,

$$\tilde{\omega}(t, \bar{X}(t, 0, \mathbf{x}), \zeta) \cong \tilde{\omega}(0, \mathbf{x}, \zeta) \qquad \text{in a weak sense}$$

Thus, we have

$$\tilde{\mathbf{u}}(t, \bar{X}(t, 0, \mathbf{x}), \zeta) \cong \int_\Omega S(\bar{X}, \mathbf{x}'')\, \tilde{\omega}(t, \mathbf{x}'', \zeta)\, d\mathbf{x}''$$

and the change of variable $\mathbf{x}'' = \bar{X}(t, 0, \mathbf{x}')$ gives

$$\tilde{\mathbf{u}}(t, \bar{X}(t, 0, \mathbf{x}), \zeta) \cong \int_\Omega S(\bar{X}(t, 0, \mathbf{x}), \bar{X}(t, 0, \mathbf{x}'))\, \tilde{\omega}(0, \mathbf{x}', \zeta)\, d\mathbf{x}'$$

But for $t \leqslant \tau$ we have

$$\bar{X}(t, 0, \mathbf{x}) = \mathbf{x} + t\bar{\mathbf{u}}(0, \mathbf{x}) + \cdots$$

and

$$S(\bar{X}(t, 0, \mathbf{x}), \bar{X}(t, 0, \mathbf{x}')) = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left(\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} + \Theta(t, \mathbf{x}, \mathbf{x}') \right)$$

with

$$|\Theta(t, \mathbf{x}, \mathbf{x}')| \leqslant 3Kt \ll 1$$

so that

$$\tilde{\mathbf{u}}(t, \bar{X}(t, 0, \mathbf{x}), \zeta) \cong \tilde{\mathbf{u}}(0, \mathbf{x}, \zeta)$$

from which one finds

$$D_{ij}(t, \mathbf{x}) \cong -\frac{\tau}{2} \left\langle \tilde{u}_i(t, \mathbf{x}, \zeta) \, \tilde{u}_j(t, \mathbf{x}, \zeta) \right\rangle$$

and finally

$$D_{ij}(t, x) \underset{(\varepsilon \to 0)}{\approx} - (\langle \omega^2 \rangle - \bar{\omega}^2) \frac{\tau \varepsilon^2}{8\pi} \mathrm{Ln} \frac{r}{\varepsilon} \delta_{ij}$$

APPENDIX B. THE VARIATIONAL PROBLEM (VP.1)

We present here the mathematical results used in Section 3. For the sake of brevity we do not provide the complete proofs, but only outline the main steps leading to the results.

In what follows we shall assume that $p_i(\mathbf{x})$, $i = 1,..., n$, are measurable functions on Ω satisfying $0 < \eta \leqslant p_i(\mathbf{x}) \leqslant 1$ and ∇p_i square-integrable on Ω.

We denote $\mathscr{H}$ the space of $(\mathbf{J}_1,..., \mathbf{J}_n)$, where each $\mathbf{J}_i$ is a square-integrable vector function on $\Omega[\mathbf{J}_i \in L^2(\Omega; \mathbb{R}^2)]$. $\mathscr{H}$ is endowed with the Hilbert norm:

$$|(\mathbf{J}_1,..., \mathbf{J}_n)| = \left(\int_\Omega \sum_i \frac{\mathbf{J}_i^2}{p_i} d\mathbf{x} \right)^{1/2}$$

We shall use the following lemma (the proof of which is an easy exercise).

Lemma B.1.   Let $C(\mathbf{x})$ be any given nonnegative integrable function on Ω. The set of the $(\mathbf{J}_1,..., \mathbf{J}_n)$ satisfying

$$\frac{1}{2} \sum_i \frac{\mathbf{J}_i^2(\mathbf{x})}{p_i(\mathbf{x})} \leqslant C(\mathbf{x}) \qquad \text{a.e.} \quad \text{(almost everywhere) in } \Omega$$

is a weakly compact subset of $\mathscr{H}$.

As a consequence, the variational problem: $\dot{S}(\mathbf{J}_1,..., \mathbf{J}_n) = \max$ $\dot{S}(\mathscr{J}_1,..., \mathscr{J}_n)$ under the constraints

(C1) $\sum \mathscr{J}_i = 0$

(C2) $\int_\Omega \nabla\psi \cdot (\sum a_i \mathcal{J}_i)\, d\mathbf{x} = 0$

(C3) $\frac{1}{2}\sum \mathcal{J}_i^2(\mathbf{x})/p_i(\mathbf{x}) \leqslant C(\mathbf{x})$

has a solution, since the subset of $\mathcal{H}$ defined by the constraints (C1)–(C3) is weakly compact and nonempty; moreover

$$\dot{S}(\mathcal{J}_1,...,\mathcal{J}_n) = -\int_\Omega \sum_i \frac{\nabla p_i}{p_i} \cdot \mathcal{J}_i\, d\mathbf{x}$$

is a continuous linear form on $\mathcal{H}$.

Such a solution is not necessarily unique. But, by a classical theorem in functional analysis[5] the maximum value is reached at an extremal point of the convex weakly compact set defined by the above constraints. Now, one can check that any extremal point of this convex set satisfies (C3) with equality (which we shall denote (C3′).

Let us then consider a solution of the variational problem satisfying constraints (C1), (C2), (C3′). We can deduce, using a generalized version of Lagrange multipliers (which deals with an infinite number of equality constraints), that there is a constant β and a measurable function $\lambda(\mathbf{x})$ such that

$$\lambda(\mathbf{x})\,\mathbf{J}_i(\mathbf{x}) = (\nabla p_i - \beta(\bar{\omega} - a_i)\,p_i\,\nabla\psi)(\mathbf{x}) \qquad \text{a.e. in } \Omega$$

To get this result, we consider a solution $\mathbf{J}_1,...,\mathbf{J}_n$ and we write

$$\dot{S}(\mathbf{J}_1 + \delta\mathbf{J}_1,...,\mathbf{J}_n + \delta\mathbf{J}_n) \leqslant \dot{S}(\mathbf{J}_1,...,\mathbf{J}_n)$$

for any variation $\delta\mathbf{J}_i$ such that:

(∗) $\sum \delta\mathbf{J}_i = 0$

(∗∗) $\int_\Omega \nabla\psi \cdot (\sum a_i\,\delta\mathbf{J}_i)\, d\mathbf{x} = 0$

(∗∗∗) $\frac{1}{2}\sum (\mathbf{J}_i + \delta\mathbf{J}_i)^2/p_i(\mathbf{x}) = C(\mathbf{x})$

Now the trick consists in writing the variation $\delta\mathbf{J}_i$ in the form of an asymptotic expansion

$$\delta\mathbf{J}_i(\mathbf{x}) = \varepsilon\mathbf{H}_i^1(\mathbf{x}) + \cdots + \varepsilon^k\mathbf{H}_i^k(\mathbf{x}) + \cdots$$

where ε is a small parameter.

This yields at the first order

$$\int_\Omega \sum_i \nabla\,\mathrm{Ln}\,p_i \cdot \mathbf{H}_i^1(\mathbf{x})\, d\mathbf{x} = 0 \qquad \text{for all}\quad \mathbf{H}_i^1(\mathbf{x}), \quad i = 1,...,n$$

satisfying

$(*)$ $\sum \mathbf{H}_i^1(\mathbf{x}) = 0$

$(**)$ $\int_\Omega \nabla\psi \cdot (\sum a_i \mathbf{H}_i^1) \, d\mathbf{x} = 0$

$(***)$ $\sum \mathbf{J}_i(\mathbf{x}) \cdot \mathbf{H}_i^1(\mathbf{x})/p_i(\mathbf{x}) = 0$ a.e. in Ω

We conclude easily by using the following lemma, the proof of which is classical.

Lemma B.2. Let $\mathbf{F}(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$, $\mathbf{E}^1(\mathbf{x})$,..., $\mathbf{E}^N(\mathbf{x})$ be given functions in $L^2(\Omega; \mathbb{R}^d)$.

Let us assume that $\int_\Omega \mathbf{F}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \, d\mathbf{x} = 0$ for all $\mathbf{H}$ in $L^2(\Omega; \mathbb{R}^d)$ such that

$$\mathbf{H}(\mathbf{x}) \cdot \mathbf{E}^i(\mathbf{x}) = 0 \qquad \text{a.e. in } \Omega, \quad i = 1,..., N$$

$$\int_\Omega \mathbf{G}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \, d\mathbf{x} = 0$$

Then there are a real parameter β and $\lambda_1(\mathbf{x})$,..., $\lambda_N(\mathbf{x})$, measurable real functions on Ω, such that

$$\mathbf{F}(\mathbf{x}) = \beta \mathbf{G}(\mathbf{x}) + \sum_{i=1}^{N} \lambda_i(\mathbf{x}) \, \mathbf{E}^i(\mathbf{x}) \qquad \text{a.e. in } \Omega$$

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